AA214A PROJECT #6: COMPUTATION OF LINEARIZED BAROTROPIC FLUID IN A NONROTATING FRAME

Reference: G. Fischer, *A Survey of Finite-Difference Approximations to the Primitive Equations*, **Monthly Weather Review**, Vol. 93, No.1, January 1965.
The primitive equations for a linearized system, in one space variable, for a barotropic atmosphere on a non-rotating earth are employed. The basic differential equations are

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \gamma\frac{\partial p}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2} \tag{1}$$

$$\frac{\partial p}{\partial t} + \gamma\frac{\partial u}{\partial x} + U\frac{\partial p}{\partial x} = \mu\frac{\partial^2 p}{\partial x^2} \tag{2}$$

or

$$\partial_t\vec{Q} + A\partial_x\vec{Q} = \mu\partial_{xx}\vec{Q} \tag{3}$$

with

$$\vec{Q} = \begin{pmatrix} u \\ p \end{pmatrix}, \quad A = \begin{pmatrix} U & \gamma \\ \gamma & U \end{pmatrix} \tag{4}$$

Physically u should be looked upon as being the velocity disturbancce superimposed on a constant basic flow U, and p as being proportional to the depth of the fluid. The phase velocity of gravity waves is γ and μ the coefficient of lateral diffusion. The terms with coefficient U describe the advection of the quantities u and p due to the basic flow, while the terms with coefficient γ define the local changes which occur due to the presence of gravity waves, with the μ coefficient terms providing dissipation due to friction. Although it is physically incorrect to add a viscous term to the second equation, this was done to gain symmetry.
Take as the conditions: $U = 5 \times 10^3 \frac{cm}{sec}$, $\gamma = 3 \times 10^4 \frac{cm}{sec}$, and $\mu = 10^9 \frac{cm^2}{sec}$. Now one can fix the wavelength L of disturbances and specifiy a number of grid points in x, $JMAX$, which gives us a $\Delta x = \frac{L}{JMAX}$. Alternatively, we can fix $\Delta x = 2 \times 10^7 cm$ and vary the number of point $JMAX$ to study the effect of methods on various wavelenths $L = JMAX \times \Delta x$.
The flow is assumed to be periodic in x, i.e., $u(x+L,t) = u(x,t), \quad p(x+L,t) = p(x+L)$.
Given initial data at $t = 0$,

$$u(x,0) = \cos\left(\frac{2\pi}{L}x\right), \quad p(x,0) = 0 \tag{5}$$

we have the exact solution

$$(u(x,t) \pm p(x,t)) = cos\left(\frac{2\pi}{L}(x - (U \pm \gamma)t)\right) e^{\frac{-4\pi^2}{L^2}\mu t} \tag{6}$$

from which one can find u and p.

Project 6: Apply Euler Explicit OΔE Method in t with Flux Splitting in x.
Apply Euler explicit differencing in t

$$\vec{Q}^{(n+1)} = \vec{Q}^{(n)} + \Delta t \partial_t \vec{Q}^{(n)} \tag{7}$$

Now from Eq. 3

$$\partial_t \vec{Q} = -A \partial_x \vec{Q} + \mu \partial_{xx} \vec{Q} \tag{8}$$

and therefore Eq. 7 can be written as

$$\vec{Q}^{(n+1)} = \vec{Q}^{(n)} - \Delta t A \partial_x \vec{Q}^{(n)} + \Delta t \mu \partial_{xx} \vec{Q}^{(n)} \tag{9}$$

The matrix A can be $\pm$ flux split into A^+ and A^- as discussed in class. This produces the new system to be solved.

$$\vec{Q}_j^{(n+1)} = \vec{Q}_j^{(n)} - \Delta t A^+ \partial_x^b \vec{Q}_j^{(n)} - \Delta t A^- \partial_x^f \vec{Q}_j^{(n)} + \Delta t \mu \partial_{xx} \vec{Q}_j^n \tag{10}$$

where ∂_x^b is a backward differencing operator and ∂_x^f is a forward differencing operator.

Assignment

1. Derive the flux splitting equations for Eq. 10. I suggest the eigenvector matrix

$$X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

2. Program Eq. 10 for the fixed wavelengths $L = 2 \times 10^8 cm$ and $L = 4 \times 10^8 cm$ using $JMAX = 10$ and $JMAX = 20$ respectively. Use the initial condition Eq. 5. Integrate the equations using $\Delta t = 4 \times 10^2 sec$ for a total time $T = 4 \times 10^6$.

   Apply 1^{st} order backward/forward differences in x.

$$\partial_x^b u_j = \frac{u_j - u_{j-1}}{\Delta x}, \quad \partial_x^f u_j = \frac{u_{j+1} - u_j}{\Delta x}$$

   2^{nd} order accurate formulas

$$\partial_x^b u_j = \frac{3u_j - 4u_{j-1} + u_{j-2}}{2\Delta x}, \quad \partial_x^f u_j = \frac{-u_{j+2} + 4u_{j+1} - 3u_j}{2\Delta x}$$

   and 3^{rd} order accurate formulas

$$\partial_x^b u_j = \frac{2u_{j+1} + 3u_j - 6u_{j-1} + u_{j-2}}{6\Delta x}, \quad \partial_x^f u_j = \frac{-u_{j+2} + 6u_{j+1} - 3u_j - 2u_{j-1}}{6\Delta x}$$

   Hint: Look at my sample code for an easy way to handle periodic indices. For example, in MATLAB let J = [1,JMAX];, let JP = J+1;, and redefine JP(JMAX) = 1;, then JP = [2,3,.....,JMAX,1]; and u(JP(JMAX)) automatically grabs u(1)

3. Apply a different marching scheme in t. That is, replace the Euler Explicit scheme in t with another one. I suggest RK4 or Leapfrog.

4. Plot and examine comparisons of the exact and numerical solution as a function of time t at various points in x. In particular, $x = 0$.

5. Find the expression for the spatial accuracy of this method, i.e., what is er_t or what is the modified wave number for the various space differences used?

6. Find the expression for the σ root for the methods you use.

7. Derive the numerical stability condition for this system. You should get something like, numerical stability requires that

$$CFL = (\frac{\Delta t}{\Delta x})^2(|U| + \gamma)^2 + \alpha\mu\frac{\Delta t}{\Delta x^2} \leq constant \tag{11}$$

   with α a constant.

8. Study various Δt and Δx ratios, and remembering the numerical stability condition!

Suggestions, Questions:

1. What happens to the error in phase and amplitude as you refine the mesh and time step?

2. You should play around with the ratio $\frac{\Delta t}{\Delta x}$ in the CFL definition. Why? Is there an optimal value of CFL for accurate results?

3. What happens when you violate the stability condition? Try it.

General Instructions:
Follow the instructions given above and address each of the assignments. You will need to provide me with a **short** writeup of what you have done, along with some results and figures. This can be handwritten, but I prefer TeX, LaTeX or some other word processor form. Perform all the computations using MATLAB. I will also want copies of all the source codes. (You will be required to email them to me, I will make arrangements). There will be 10 minutes allotted for a short presentation in class on what you have accomplished. You should focus on the interesting aspects of your project.

1. Grading will be broken down as follows:
   (a) Writeup: 30 points.
   (b) Code: 10 points.
   (c) Presentation: 10 points.

2. This will account for 50% of your grade.

3. The other 50% comes from 25 points for turned in homework and 25 points for the midterm.